

Log-behavior of two sequences related to the elliptic integrals

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Abstract. Two interesting sequences arose in the study of the series expansions of the complete elliptic integrals, which are called the Catalan-Larcombe-French sequence $\{P_n\}_{n \geq 0}$ and the Fennessey-Larcombe-French sequence $\{V_n\}_{n \geq 0}$ respectively. In this paper, we prove the ratio log-concavity of $\{P_n\}_{n \geq 0}$, the log-convexity of $\{V_n^2 - V_{n-1}V_{n+1}\}_{n \geq 2}$, the ratio log-convexity of $\{V_n\}_{n \geq 1}$, and the log-convexity of $\{n!V_n\}_{n \geq 1}$.

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1 Introduction

Recently, there is a rising interest in the study of the log-behavior of the following two sequences defined by

$$n^2 P_n = 8(3n^2 - 3n + 1)P_{n-1} - 128(n-1)^2 P_{n-2}, \quad (1.1)$$

$$(n-1)n^2 V_n = 8(n-1)(3n^2 - n - 1)V_{n-1} - 128(n-2)n^2 V_{n-2}, \quad (1.2)$$

with the initial values $P_0 = V_0 = 1$ and $P_1 = V_1 = 8$. The sequences $\{P_n\}_{n \geq 0}$ and $\{V_n\}_{n \geq 0}$ are known as the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence, respectively. They arise naturally from the series expansions of the complete elliptic integrals, see [2, 7–10]. The main objective of this paper is to prove the ratio log-concavity of $\{P_n\}_{n \geq 0}$, the log-convexity of $\{V_n^2 - V_{n-1}V_{n+1}\}_{n \geq 2}$, the ratio log-convexity of $\{V_n\}_{n \geq 1}$, and the log-convexity of $\{n!V_n\}_{n \geq 1}$. Let us first review some background.

Recall that a real sequence $\{S_n\}_{n \geq 0}$ is said to be log-concave (resp. log-convex) if $S_n^2 \geq S_{n-1}S_{n+1}$ (resp. $S_n^2 \leq S_{n-1}S_{n+1}$) for all $n \geq 1$, and it is said to be strictly log-concave (resp. strictly log-convex) if the inequality is strict. Let \mathcal{L} be an operator on $\{S_n\}_{n \geq 0}$ such that

$$\mathcal{L}(\{S_n\}_{n \geq 0}) = \{S_{n-1}S_{n+1} - S_n^2\}_{n \geq 1}.$$

The sequence $\{S_n\}_{n \geq 0}$ is called k -log-convex if $\mathcal{L}^i(\{S_n\}_{n \geq 0})$ is log-convex for $0 \leq i \leq k-1$, and $\{S_n\}_{n \geq 0}$ is called infinitely log-convex if $\mathcal{L}^k(\{S_n\}_{n \geq 0})$ is log-convex for any $k \geq 1$, see Chen and Xia [4]. A real sequence $\{S_n\}_{n \geq 0}$ is called ratio log-concave (resp. ratio log-convex) if the sequence $\{S_n/S_{n-1}\}_{n \geq 1}$ is log-concave (resp. log-convex), see Chen, Guo and Wang [3]. A real sequence $\{S_n\}_{n \geq 0}$ is called log-balanced if $\{S_n\}_{n \geq 0}$ is log-convex while $\{S_n/n!\}_{n \geq 0}$ is log-concave, see Došlić [5].

The log-convexity of $\{P_n\}_{n \geq 0}$, conjectured by Sun [12], has been proved by Xia and Yao [13] and independently by Zhao [16]. The log-concavity of $\{V_n\}_{n \geq 1}$, conjectured by Zhao [17], has been confirmed by Yang and Zhao [15]. The 2-log-convexity of $\{P_n\}_{n \geq 0}$ has been shown by Sun and Wu [11]. It is natural to consider whether $\{V_n\}_{n \geq 1}$ is 2-log-concave or not. The first main result of this paper gives the answer.

Theorem 1.1. *The sequence $\{V_n^2 - V_{n-1}V_{n+1}\}_{n \geq 2}$ is strictly log-convex, that is, for $n \geq 3$,*

$$(V_n^2 - V_{n-1}V_{n+1})^2 < (V_{n-1}^2 - V_{n-2}V_n)(V_{n+1}^2 - V_nV_{n+2}). \quad (1.3)$$

Note that Theorem 1.1 does not imply the 2-log-convexity of $\{V_n\}_{n \geq 1}$, since $\{V_n\}_{n \geq 1}$ itself is log-concave. Chen, Guo and Wang showed that the ratio log-concavity (resp. ratio log-convexity) of a sequence $\{S_n\}_{n \geq N}$ implies the strict log-concavity (resp. strict log-convexity) of the sequence $\{\sqrt[n]{S_n}\}_{n \geq N}$ under an initial condition [3, Theorems 3.1 & 3.6]. Although the strictly log-concavity of $\{\sqrt[n]{P_n}\}_{n \geq 1}$ and $\{\sqrt[n]{V_n}\}_{n \geq 1}$ had been proved by Zhao [18] in a direct way, the ratio log-behavior of $\{P_n\}_{n \geq 0}$ and $\{V_n\}_{n \geq 1}$ still deserve attention, and are precisely described as follows.

Theorem 1.2. *The sequence $\{P_n\}_{n \geq 0}$ is ratio log-concave, that is, for $n \geq 2$,*

$$(P_n/P_{n-1})^2 \geq (P_{n-1}/P_{n-2})(P_{n+1}/P_n). \quad (1.4)$$

Theorem 1.3. *The sequence $\{V_n\}_{n \geq 1}$ is ratio log-convex, that is, for $n \geq 3$,*

$$(V_n/V_{n-1})^2 \leq (V_{n-1}/V_{n-2})(V_{n+1}/V_n). \quad (1.5)$$

Notice that the strictly log-concavity of $\{\sqrt[n]{P_n}\}_{n \geq 1}$ is a consequence of the criterion [3, Theorem 3.1] and Theorem 1.2, while the strictly log-concavity of $\{\sqrt[n]{V_n}\}_{n \geq 1}$ can not be obtained from the criterion [3, Theorem 3.1] and Theorem 1.3. By further study, we also prove the log-convexity of the sequence $\{n!V_n\}_{n \geq 0}$.

Theorem 1.4. *The sequence $\{n!V_n\}_{n \geq 0}$ is strictly log-convex, that is, for $n \geq 1$,*

$$nV_n^2 < (n+1)V_{n-1}V_{n+1}. \quad (1.6)$$

Since $\{V_n\}_{n \geq 1}$ is log-concave, it follows that the sequence $\{n!V_n\}_{n \geq 1}$ is log-balanced by Theorem 1.4. We notice that Došlić's criterion of determining log-balancedness [5, Proposition 3.4] is not available for the sequence $\{n!V_n\}_{n \geq 1}$. It should be mentioned that

Bender and Canfield had given a different criterion [1, Theorem 1] for determining log-balancedness of $\{n!S_n\}_{n \geq 1}$, which also does not apply to $\{n!V_n\}_{n \geq 1}$, although they did not name the concept of log-balancedness.

This paper is organized as follows. In Section 2, we prove lower bounds and upper bounds for the ratios V_n/V_{n-1} and P_n/P_{n-1} based on their three-term recurrence relations. These bounds will be used in the proofs of our main results. In Section 3, we prove Theorem 1.1 by establishing a criterion, which slightly modifies that of Chen and Xia [4, Theorem 2.1]. In Section 4, we give the proofs of Theorem 1.2 and Theorem 1.3 by building two criteria along with the spirit showed in Chen, Guo and Wang [3, §4]. In Section 5, we complete the proof of Theorem 1.4. We conclude this paper with a few conjectures on log-behavior related to the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence.

2 Bounds for V_n/V_{n-1} and P_n/P_{n-1}

In this section we establish two sets of bounds, one for the ratio V_n/V_{n-1} and the other for the ratio P_n/P_{n-1} , that will lead to our main results. These bounds are obtained by the heuristic approach given by Chen and Xia [4]. Since three of our main results are related to V_n , we first consider the bounds of V_n/V_{n-1} . For $n \geq 1$, let

$$s(n) = \frac{16(n^5 + n^2 + 3n + 12)}{n^5}, \quad \text{and} \quad t(n) = \frac{16(n+1)}{n}. \quad (2.1)$$

Lemma 2.1. *Let $s(n)$ and $t(n)$ be given by (2.1). Then for all integers $n \geq 6$, we have*

$$s(n) < \frac{V_n}{V_{n-1}} < t(n). \quad (2.2)$$

Proof. For notational convenience, let $r(n) = V_n/V_{n-1}$, and We first prove $r(n) > s(n)$ for $n \geq 6$ by using mathematical induction on n . By the recurrence relation (1.2), we have

$$r(n+1) = \frac{8(3n^2 + 5n + 1)}{(n+1)^2} - \frac{128(n-1)}{nr(n)}, \quad n \geq 1, \quad (2.3)$$

with the initial value $r(1) = 8$. It is easily checked that $r(6) = 20482/1269 > 1307/81 = s(6)$ by (2.3) and (2.1). Assume $r(n) > s(n)$ holds for $n \geq 6$, and we proceed to show that $r(n+1) > s(n+1)$. Note that

$$\begin{aligned} r(n+1) - s(n+1) &= \frac{8(3n^2 + 5n + 1)}{(n+1)^2} - \frac{128(n-1)}{nr(n)} \\ &\quad - \frac{16(n^5 + 5n^4 + 10n^3 + 11n^2 + 10n + 17)}{(n+1)^5} \\ &= \frac{8n(n^5 + 4n^4 + 5n^3 - n^2 - 12n - 33)r(n) - 128(n-1)(n+1)^5}{n(n+1)^5r(n)}. \end{aligned}$$

Clearly, $n^5 + 4n^4 + 5n^3 - n^2 - 12n - 33 = (n^3 - 1)(n^2 + 4n + 5) - 8n - 28 > 0$ for $n \geq 2$ and $r(n) > 0$ for $n \geq 1$. By the induction hypothesis, we have $r(n) > s(n)$. Thus for $n \geq 6$, it follows that

$$\begin{aligned} r(n+1) - s(n+1) &> \frac{8n(n^5 + 4n^4 + 5n^3 - n^2 - 12n - 33)s(n) - 128(n-1)(n+1)^5}{n(n+1)^5 r(n)} \\ &= \frac{1152(7n^4 + 5n^3 - 9n^2 - 27n - 44)}{n^5(n+1)^5 r(n)}, \end{aligned}$$

which is clearly positive for $n \geq 6$ since $7n^4 + 5n^3 - 9n^2 - 27n - 44 = (n+2)(n^2 + n + 3)(7n - 16) + 4n^2 + 11n + 52 > 0$ for $n \geq 6$. This proves $r(n) > s(n)$ for $n \geq 6$.

For $n \geq 6$, the detailed proof of the inequality $r(n) < t(n)$ are similar to that of $r(n) > s(n)$, and hence is omitted here. \square

We now present a lower bound and an upper bound of P_n/P_{n-1} . For $n \geq 1$, let

$$l(n) = \frac{24(3n^2 - 3n + 1)}{5n^2}, \quad \text{and} \quad \ell(n) = \frac{16(n^3 - n^2 - 1)}{n^3}. \quad (2.4)$$

Lemma 2.2. *Let $l(n)$ and $\ell(n)$ be given by (2.4), then for all integers $n \geq 6$, we have*

$$l(n) < \frac{P_n}{P_{n-1}} < \ell(n). \quad (2.5)$$

Proof. By using mathematical induction on n , it is easy to show that $P_n/P_{n-1} > l(n)$ for $n \geq 1$, and $P_n/P_{n-1} < \ell(n)$ for $n \geq 6$. The detailed proof is similar to that of Lemma 2.1, and hence is omitted here. \square

Remark 2.3. *Notice that Hou and Zhang [6] have established an asymptotic method to prove k -log-convexity of some sequences except for certain terms at the beginning, and they obtained the bounds by a computer algorithm.*

3 Proof of Theorem 1.1

In this section, we show the proof of Theorem 1.1 by presenting a criterion for determining the log-convexity of the sequence $\{S_n^2 - S_{n-1}S_{n+1}\}$, where $\{S_n\}_{n \geq 0}$ is a positive sequence that satisfies the recurrence

$$S_n = a(n)S_{n-1} + b(n)S_{n-2}, \quad n \geq 2, \quad (3.1)$$

with real $a(n)$ and $b(n)$. Our criterion slightly modifies that of Chen and Xia [4, Theorem 2.1]. We notice that in the criterion of Chen and Xia, the sequence $\{S_n\}_{n \geq 0}$ is assumed to be log-convex and an upper bound for S_n/S_{n-1} subject to certain conditions is also needed, while these constraints are not required in ours.

Theorem 3.1. For a positive sequence $\{S_n\}_{n \geq 0}$ satisfying the relation (3.1), let

$$\begin{aligned}
c_0(n) &= -b^2(n+1)[a^2(n+2) + b(n+1) - a(n+2)a(n+3) - b(n+3)]; \\
c_1(n) &= b(n+1)[2a(n+2)b(n+1) + 2a(n+3)a(n+2)a(n+1) \\
&\quad + a(n+3)b(n+2) + 2a(n+1)b(n+3) - 2a^2(n+2)a(n+1) \\
&\quad - 2a(n+2)b(n+2) - 3a(n+1)b(n+1)]; \\
c_2(n) &= 4a(n+1)a(n+2)b(n+1) + 2b(n+1)b(n+2) + a^2(n+1)a(n+2)a(n+3) \\
&\quad + a(n+1)a(n+3)b(n+2) + a^2(n+1)b(n+3) - 3a^2(n+1)b(n+1) \\
&\quad - a(n+3)a(n+2)b(n+1) - a^2(n+2)a^2(n+1) - b(n+3)b(n+1) \\
&\quad - 2a(n+2)a(n+1)b(n+2) - b^2(n+2); \\
c_3(n) &= 2a^2(n+1)a(n+2) + 2a(n+1)b(n+2) - a(n+1)b(n+3) - a^3(n+1) \\
&\quad - a(n+1)a(n+2)a(n+3) - a(n+3)b(n+2);
\end{aligned}$$

and

$$\Delta(n) = 4c_2^2(n) - 12c_1(n)c_3(n).$$

Suppose that $c_3(n) > 0$ and $\Delta(n) > 0$ for all $n \geq N$, where N is a positive integer. If there exists $f(n)$ such that for all $n \geq N$,

$$(I) \quad \frac{S_n}{S_{n-1}} \geq f(n);$$

$$(II) \quad f(n) \geq \frac{-2c_2(n) + \sqrt{\Delta(n)}}{6c_3(n)};$$

$$(III) \quad c_3(n)f(n)^3 + c_2(n)f(n)^2 + c_1(n)f(n) + c_0(n) > 0,$$

then the sequence $\{S_n^2 - S_{n-1}S_{n+1}\}_{n \geq N}$ is strictly log-convex, that is, for $n \geq N$,

$$(S_{n+1}^2 - S_n S_{n+2})^2 < (S_n^2 - S_{n-1} S_{n+1})(S_{n+2}^2 - S_{n+1} S_{n+3}). \quad (3.2)$$

Proof. By the recurrence relation (3.1) and the positivity of the sequence $\{S_n\}_{n \geq 0}$, for $n \geq N$, we have

$$\begin{aligned}
&(S_n^2 - S_{n-1}S_{n+1})(S_{n+2}^2 - S_{n+1}S_{n+3}) - (S_{n+1}^2 - S_n S_{n+2})^2 \\
&= S_{n+1}(2S_n S_{n+1} S_{n+2} + S_{n-1} S_{n+1} S_{n+3} - S_{n+1}^3 - S_n^2 S_{n+3} - S_{n-1} S_{n+2}^2) \\
&= S_{n+1}(c_3(n)S_n^3 + c_2(n)S_n^2 S_{n-1} + c_1(n)S_n S_{n-1}^2 + c_0(n)S_{n-1}^3) \\
&= \frac{S_{n+1}}{S_{n-1}^3} \left[c_3(n) \left(\frac{S_n}{S_{n-1}} \right)^3 + c_2(n) \left(\frac{S_n}{S_{n-1}} \right)^2 + c_1(n) \left(\frac{S_n}{S_{n-1}} \right) + c_0(n) \right].
\end{aligned}$$

In order to prove (3.2), it is sufficient to show that for $n \geq N$,

$$c_3(n) \left(\frac{S_n}{S_{n-1}} \right)^3 + c_2(n) \left(\frac{S_n}{S_{n-1}} \right)^2 + c_1(n) \left(\frac{S_n}{S_{n-1}} \right) + c_0(n) > 0. \quad (3.3)$$

Let us consider the polynomial $w(x) = c_3(n)x^3 + c_2(n)x^2 + c_1(n)x + c_0(n)$. Observe that

$$w'(x) = 3c_3(n)x^2 + 2c_2(n)x + c_1(n).$$

Since $c_3(n) > 0$ and $\Delta(n) > 0$ for all $n \geq N$, we have the quadratic function $w'(x) \geq 0$ for $x \geq \frac{-2c_2(n) + \sqrt{\Delta(n)}}{6c_3(n)}$, which means that $w(x)$ is increasing for $x \in [\frac{-2c_2(n) + \sqrt{\Delta(n)}}{6c_3(n)}, +\infty)$. By conditions (I) and (II), we have $\frac{S_n}{S_{n-1}} \geq f(n) \geq \frac{-2c_2(n) + \sqrt{\Delta(n)}}{6c_3(n)}$, it follows that for $n \geq N$,

$$w\left(\frac{S_n}{S_{n-1}}\right) \geq w(f(n)).$$

By condition (III), we have $w(f(n)) > 0$ for any $n \geq N$. Thus we have $w\left(\frac{S_n}{S_{n-1}}\right) > 0$ for $n \geq N$, which leads to (3.3). This completes the proof. \square

By the proof of Theorem 3.1, one can easily conclude that $\{S_n^2 - S_{n-1}S_{n+1}\}_{n \geq N}$ is log-convex if there exists a positive integer N such that for all $n \geq N$, $c_3(n) > 0$, $\Delta(n) < 0$, and the conditions (I) and (III) in Theorem 3.1 holds.

We are now ready to prove Theorem 1.1

Proof of Theorem 1.1. It is easy to verify that (1.3) is true for $n = 3, 4, 5$. We aim to prove (1.3) for $n \geq 6$ by applying Theorem 3.1, that is, for $n \geq 6$,

$$(V_n^2 - V_{n-1}V_{n+1})^2 < (V_{n-1}^2 - V_{n-2}V_n)(V_{n+1}^2 - V_nV_{n+2}).$$

Compare (1.2) and (3.1), we have

$$V_n = a(n)V_{n-1} + b(n)V_{n-2}, \quad n \geq 2,$$

where

$$a(n) = \frac{8(3n^2 - n - 1)}{n^2}, \quad b(n) = -\frac{128(n - 2)}{n - 1}. \quad (3.4)$$

To apply Theorem 1.1, we first verify that $c_3(n) > 0$ and $\Delta(n) > 0$ for $n \geq 1$. By computing, it follows that

$$c_3(n) = \frac{512(n^8 + 17n^7 + 131n^6 + 484n^5 + 872n^4 + 682n^3 + 51n^2 - 177n - 45)}{(n+1)^6(n+2)^2(n+3)^2},$$

and

$$\begin{aligned} \Delta(n) = & \frac{67108864}{(n+1)^8(n+2)^8(n+3)^4n^2} (n^{18} + 40n^{17} + 752n^{16} + 8732n^{15} + 69566n^{14} \\ & + 399108n^{13} + 1687512n^{12} + 5311376n^{11} + 12451223n^{10} + 21531796n^9 \\ & + 26834592n^8 + 23183984n^7 + 13750782n^6 + 8285676n^5 + 10267104n^4 \end{aligned}$$

$$+12477380n^3 + 9141001n^2 + 3600576n + 596160).$$

Clearly, both $c_3(n)$ and $\Delta(n)$ are positive for all $n \geq 1$.

Let $N = 6$ and $f(n) = s(n)$ for $n \geq N$ where $s(n)$ is defined in (2.1). We proceed to verify the conditions (I), (II) and (III) in Theorem 3.1. It is clear that $\frac{V_n}{V_{n-1}} \geq f(n)$ for $n \geq 6$ by Lemma 2.1, which is the condition (I). We next verify the condition (II). By computing we get

$$\begin{aligned} & [6c_3(n)f(n) + 2c_2(n)]^2 - \Delta(n) \\ &= \frac{805306368}{n^{10}(n+3)^4(n+2)^6(n+1)^{12}} (3n^{26} + 78n^{25} + 952n^{24} + 7054n^{23} + 37260n^{22} \\ & \quad + 172168n^{21} + 821087n^{20} + 3833124n^{19} + 15316869n^{18} + 49491792n^{17} \\ & \quad + 130518035n^{16} + 295700768n^{15} + 624334735n^{14} + 1306596402n^{13} \\ & \quad + 2645121752n^{12} + 4751027330n^{11} + 6964163254n^{10} + 7754776872n^9 \\ & \quad + 5930725839n^8 + 2290239180n^7 - 689241033n^6 - 1426673628n^5 \\ & \quad - 697884741n^4 - 39615804n^3 + 90921852n^2 + 32775840n + 3499200), \end{aligned}$$

which is easily checked to be positive for $n \geq 6$. Note that

$$\begin{aligned} & 6c_3(n)f(n) + 2c_2(n) \\ &= \frac{8192}{(n+1)^6(n+2)^4(n+3)^2n^5} (n^{15} + 22n^{14} + 235n^{13} + 1362n^{12} + 4663n^{11} \\ & \quad + 10794n^{10} + 23419n^9 + 65264n^8 + 184207n^7 + 395220n^6 + 572275n^5 \\ & \quad + 497880n^4 + 183150n^3 - 56592n^2 - 67176n - 12960), \end{aligned}$$

which is clearly positive for $n \geq 6$. Thus it follows that

$$6c_3(n)f(n) + 2c_2(n) \geq \sqrt{\Delta(n)},$$

for $n \geq 6$, which is equivalent to the condition (II).

Now it remains to verify the condition (III). To this end, we find that

$$\begin{aligned} & c_3(n)f(n)^3 + c_2(n)f(n)^2 + c_1(n)f(n) + c_0(n) \\ &= \frac{3145728}{(n+1)^6(n+2)^4(n+3)^2n^{15}} (66n^{17} + 900n^{16} + 6674n^{15} + 34000n^{14} \\ & \quad + 124157n^{13} + 336864n^{12} + 722550n^{11} + 1356276n^{10} + 2548054n^9 \\ & \quad + 4990502n^8 + 9033247n^7 + 13148436n^6 + 13877382n^5 + 9189072n^4 \\ & \quad + 2222712n^3 - 1490400n^2 - 1178496n - 207360) > 0 \end{aligned}$$

for all $n \geq 6$, which can be easily checked. This completes the proof. \square

4 Proofs of Theorem 1.2 and Theorem 1.3

In this section we give the detailed proofs of Theorem 1.2 and Theorem 1.3. Note that Chen, Guo and Wang had showed a criterion [3, Theorem 4.5] for ratio log-concavity of a sequence subject to the recurrence (3.1). But their criterion can not be applied to prove our results. Along with their spirit, we establish two criteria for ratio log-concavity and ratio log-convexity, respectively, of a sequence subject to (3.1). The first one is as follows.

Theorem 4.1. *Let $\{S_n\}_{n \geq 0}$ be a positive sequence satisfying the recurrence relation (3.1), that is,*

$$S_n = a(n)S_{n-1} + b(n)S_{n-2}, \quad n \geq 2.$$

Suppose $a(n) > 0$ and $b(n) < 0$ for $n \geq N$ where N is a nonnegative integer. If there exists two functions $u(n)$ and $v(n)$ such that for all $n \geq N + 2$,

$$(i) \quad \frac{a(n)}{2} \leq u(n) \leq \frac{S_n}{S_{n-1}} \leq v(n);$$

$$(ii) \quad 4u^3(n) - 3a(n)u^2(n) - a(n+1)b(n) \geq 0;$$

$$(iii) \quad v^4(n) - a(n)v^3(n) - a(n+1)b(n)v(n) - b(n)b(n+1) \leq 0,$$

then $\{S_n\}_{n \geq N}$ is ratio log-concave, that is, for $n \geq N + 2$,

$$(S_n/S_{n-1})^2 \geq (S_{n-1}/S_{n-2})(S_{n+1}/S_n). \quad (4.1)$$

Proof. It is clear that (4.1) can be rewritten as

$$S_n^3 S_{n-2} - S_{n-1}^3 S_{n+1} \geq 0. \quad (4.2)$$

By the recurrence relation (3.1), we have

$$\begin{aligned} & S_n^3 S_{n-2} - S_{n-1}^3 S_{n+1} \\ &= \frac{1}{b(n)} S_n^3 (S_n - a(n)S_{n-1}) - S_{n-1}^3 (a(n+1)S_n + b(n+1)S_{n-1}) \\ &= \frac{S_{n-1}^4}{b(n)} \left[\left(\frac{S_n}{S_{n-1}} \right)^4 - a(n) \left(\frac{S_n}{S_{n-1}} \right)^3 - a(n+1)b(n) \left(\frac{S_n}{S_{n-1}} \right) - b(n)b(n+1) \right]. \end{aligned}$$

Note that $b(n) < 0$ for $n \geq N + 2$. In order to prove (4.2), it suffices to verify that

$$\left(\frac{S_n}{S_{n-1}} \right)^4 - a(n) \left(\frac{S_n}{S_{n-1}} \right)^3 - a(n+1)b(n) \left(\frac{S_n}{S_{n-1}} \right) - b(n)b(n+1) \leq 0, \quad (4.3)$$

for $n \geq N + 2$. Define

$$h(x) = x^4 - a(n)x^3 - a(n+1)b(n)x - b(n)b(n+1).$$

Then (4.3) is equivalent to

$$h\left(\frac{S_n}{S_{n-1}}\right) \leq 0,$$

for $n \geq N + 2$. Observe that

$$h'(x) = 4x^3 - 3a(n)x^2 - a(n+1)b(n),$$

and

$$h''(x) = 12x^2 - 6a(n)x.$$

Since $a(n) > 0$, $h''(x) \geq 0$ for $x \geq a(n)/2$, which implies that $h'(x)$ is increasing for $x \geq a(n)/2$. Note that $u(n) \geq a(n)/2$ by the condition (i). Then we have $h'(x) \geq h'(u(n))$ for $x \geq u(n)$. By the condition (ii), we have $h'(u(n)) \geq 0$. It follows that $h'(x) \geq 0$ for $x \geq u(n)$, and hence $h(x)$ is increasing for $x \geq u(n)$. Then we have $h(S_n/S_{n-1}) \leq h(v(n))$ since $u(n) \leq S_n/S_{n-1} \leq v(n)$ by the condition (i). Now it remains to show that $h(v(n)) \leq 0$, which is the condition (iii). This completes the proof. \square

The criterion for the ratio log-convexity of a sequence subject to (3.1) is as follows.

Theorem 4.2. *Let $\{S_n\}_{n \geq 0}$ be a positive sequence satisfying the recurrence relation (3.1). Suppose $a(n) > 0$ and $b(n) < 0$ for $n \geq N$ where N is a nonnegative integer. If there exists a function $g(n)$ such that for all $n \geq N + 2$,*

$$(i') \quad \frac{a(n)}{2} \leq g(n) \leq \frac{S_n}{S_{n-1}};$$

$$(ii') \quad 4g^3(n) - 3a(n)g^2(n) - a(n+1)b(n) \geq 0;$$

$$(iii') \quad g^4(n) - a(n)g^3(n) - a(n+1)b(n)g(n) - b(n)b(n+1) \geq 0,$$

then $\{S_n\}_{n \geq N}$ is ratio log-convex, that is, for $n \geq N + 2$,

$$(S_n/S_{n-1})^2 \leq (S_{n-1}/S_{n-2})(S_{n+1}/S_n). \quad (4.4)$$

Proof. The detailed proof of Theorem 4.2 is similar to that of Theorem 4.1, and hence is omitted here. \square

With the help of Theorem 4.1, we are ready to show the proof of Theorem 1.2.

Proof of Theorem 1.2. It is easy to verify that (1.4) holds for $2 \leq n \leq 5$. We aim to prove (1.4) for $n \geq 6$, by applying Theorem 4.1. Compare (1.1) and (3.1), we have

$$P_n = a(n)P_{n-1} + b(n)P_{n-2}$$

for $n \geq 2$, where

$$a(n) = \frac{8(3n^2 - 3n + 1)}{n^2}, \quad b(n) = -\frac{128(n-1)^2}{n^2}.$$

Set $N = 4$. Clearly, $a(n) > 0$ and $b(n) < 0$ for $n \geq 6$. It suffices to verify the conditions (i), (ii) and (iii) in Theorem 4.1. To this end, let $u(n) = l(n)$ and $v(n) = \ell(n)$ where $l(n)$ and $\ell(n)$ are given by (2.4). Note that $u(n) = 3a(n)/5 > a(n)/2$. By Lemma 2.2, we have $u(n) \leq S_n/S_{n-1} \leq v(n)$ for $n \geq 6$. This verifies the conditions in (i) of Theorem 4.1. It remains to verify the conditions (ii) and (iii) in Theorem 4.1. By computing, we obtain that

$$4u^3(n) - 3a(n)u^2(n) - a(n+1)b(n) = \frac{512 A(n)}{125n^6(n+1)^2},$$

where

$$A(n) = 21n^8 - 21n^7 + 229n^6 - 1208n^5 + 736n^4 + 486n^3 - 513n^2 + 189n - 27,$$

and

$$v^4(n) - a(n)v^3(n) - a(n+1)b(n)v(n) - b(n)b(n+1) = \frac{-16384 B(n)}{n^{12}(n+1)^2},$$

where

$$B(n) = 4n^{11} - 7n^{10} - 3n^9 - 5n^8 + 9n^7 + 20n^6 + 10n^5 - 2n^4 - 18n^3 - 18n^2 - 10n - 4.$$

It is easy to check that $A(n) > 0$ and $B(n) > 0$ for $n \geq 6$, which confirm the conditions (ii) and (iii) in Theorem 4.1. This completes the proof. \square

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. It is easy to check that (1.5) is true for $3 \leq n \leq 5$. We aim to prove (1.5) for $n \geq 6$ by using Theorem 4.2. In the proof of Theorem 1.1, we have obtained that

$$V_n = a(n)V_{n-1} + b(n)V_{n-2},$$

for $n \geq 2$, where

$$a(n) = \frac{8(3n^2 - n - 1)}{n^2}, \quad b(n) = -\frac{128(n-2)}{n-1}.$$

Let $N = 4$. Clearly, $a(n) > 0$ and $b(n) < 0$ for $n \geq 6$. It suffices to verify the conditions (i'), (ii') and (iii') in Theorem 4.2. For this purpose, let $g(n) = s(n)$ for $n \geq 6$, where $s(n)$ is defined in (2.1). First by Lemma 2.1 we have $g(n) \leq S_n/S_{n-1}$ for $n \geq 6$. Observe that

$$g(n) - \frac{a(n)}{2} = \frac{4(n^5 + n^4 + n^3 + 4n^2 + 12n + 48)}{n^5} > 0,$$

for $n \geq 1$. This confirms the condition (i') in Theorem 4.2.

It remains to verify the conditions (ii') and (iii') in Theorem 4.2. By computation we have that

$$4g^3(n) - 3a(n)g^2(n) - a(n+1)b(n) = \frac{1024 C(n)}{n^{15}(n-1)(n+1)^2},$$

where

$$C(n) = n^{18} + 3n^{17} + 5n^{16} + 12n^{15} + 48n^{14} + 222n^{13} + 342n^{12} + 300n^{11} + 960n^{10}$$

$$+ 2902n^9 + 6142n^8 + 3956n^7 - 448n^6 + 9450n^5 + 25776n^4 + 31536n^3 \\ - 5184n^2 - 48384n - 27648,$$

and

$$g^4(n) - a(n)g^3(n) - a(n+1)b(n)g(n) - b(n)b(n+1) = \frac{16384 D(n)}{n^{20}(n-1)(n+1)^2},$$

where

$$D(n) = 24n^{18} + 57n^{17} + 96n^{16} + 234n^{15} + 706n^{14} + 1908n^{13} + 2616n^{12} + 3126n^{11} \\ + 8130n^{10} + 18198n^9 + 27248n^8 + 14970n^7 + 5478n^6 + 49572n^5 + 97308n^4 \\ + 77760n^3 - 58752n^2 - 165888n - 82944.$$

It is clear that $C(n) > 0$ and $D(n) > 0$ for $n \geq 6$. Hence the conditions (ii') and (iii') in Theorem 4.2 are verified for $n \geq 6$. This completes the proof. \square

5 Proof of Theorem 1.4

In this section, we complete the proof of Theorem 1.4, the log-convexity of the sequence $\{n!V_n\}_{n \geq 0}$. To make the proof more concise, we need a modified lower bound for the ratio V_n/V_{n-1} . For $n \geq 1$, let

$$\tau(n) = \frac{16(n^3 + 1)}{n^3}. \quad (5.1)$$

Note that $s(n) - \tau(n) = 48(n+4)/n^5 > 0$ for $n \geq 1$ where $s(n)$ is given in (2.1). Let $r(n) = V_n/V_{n-1}$ and $t(n)$ be defined in (2.1). Then by Lemma 2.1 it is easy to check that

$$\tau(n) < r(n) < t(n) \quad (5.2)$$

for $n \geq 2$. With these two bounds, we are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. For $n = 1$, by the recurrence (1.2), we have $V_1^2 = 64 < 288 = 2V_0V_1$. We proceed to prove (1.6) for $n \geq 2$. Note that (1.6) can be rewritten as

$$\frac{r(n)}{r(n+1)} < \frac{n+1}{n}.$$

Since $r(n) > 0$ for $n \geq 1$, by (2.3) we obtain that for $n \geq 1$,

$$\begin{aligned} \frac{r(n)}{r(n+1)} &= \frac{n(n+1)^2 r^2(n)}{8n(3n^2 + 5n + 1)r(n) - 128(n-1)(n+1)^2} \\ &= \frac{n(n+1)^2 r(n)}{8n(3n^2 + 5n + 1) - 128(n-1)(n+1)^2/r(n)}, \end{aligned}$$

with the initial value $r(1) = 8$. Then it suffices to show that

$$\frac{n(n+1)^2 r(n)}{8n(3n^2 + 5n + 1) - 128(n-1)(n+1)^2/r(n)} < \frac{n+1}{n},$$

for $n \geq 2$. By (5.2), we conclude that

$$\begin{aligned} & \frac{n(n+1)^2 r(n)}{8n(3n^2 + 5n + 1) - 128(n-1)(n+1)^2/r(n)} - \frac{n+1}{n} \\ & \leq \frac{n(n+1)^2 t(n)}{8n(3n^2 + 5n + 1) - 128(n-1)(n+1)^2/\tau(n)} - \frac{n+1}{n} \\ & = -\frac{2n^2 + n - 1}{n(2n^4 + 2n^3 + 4n + 1)}, \end{aligned}$$

which is clearly negative for $n \geq 2$. This completes the proof. \square

We conclude this paper with a few conjectures related to the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence.

Conjecture 5.1. *The sequence $\{V_n^2 - V_{n-1}V_{n+1}\}_{n \geq 2}$ is infinitely log-convex.*

Very recently, Wang and Zhu [14] showed that Stieltjes moment sequences are infinitely log-convex. This provides a possibility for proving Conjecture 5.1 with some analysis tools.

Let \mathcal{R} be an operator on a sequence $\{S_n\}_{n \geq 0}$ such that

$$\mathcal{R}(\{S_n\}_{n \geq 0}) = \{S_{n+1}/S_n\}_{n \geq 0}.$$

Conjecture 5.2. *For all integer $k \geq 1$, the sequence $\mathcal{R}^k(\{P_n\}_{n \geq 0})$ except for the first k terms at the beginning is log-concave if k odd, and is log-convex if k even.*

Conjecture 5.3. *For all integer $k \geq 1$, the sequence $\mathcal{R}^k(\{V_n\}_{n \geq 1})$ is log-convex if k odd, and is log-concave if k even.*

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